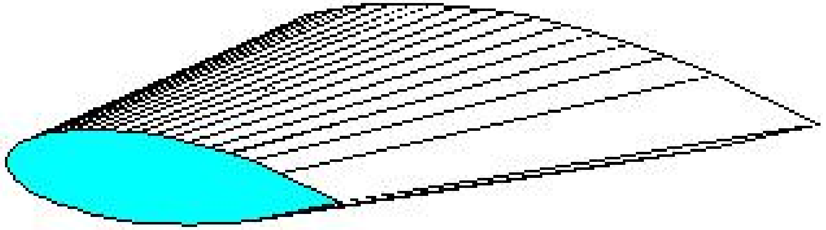
《数值分析》14

主要内容：

- 样条插值的算例
- 三次样条的概念
- 用一阶导数表示的样条
- 三次样条的极性

例1. 飞机机翼剖面图

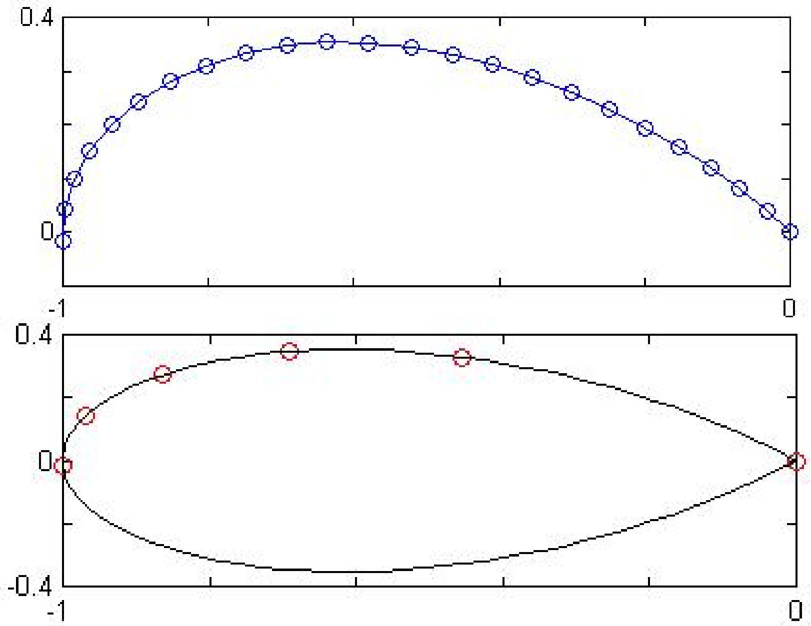


1. 数据采集

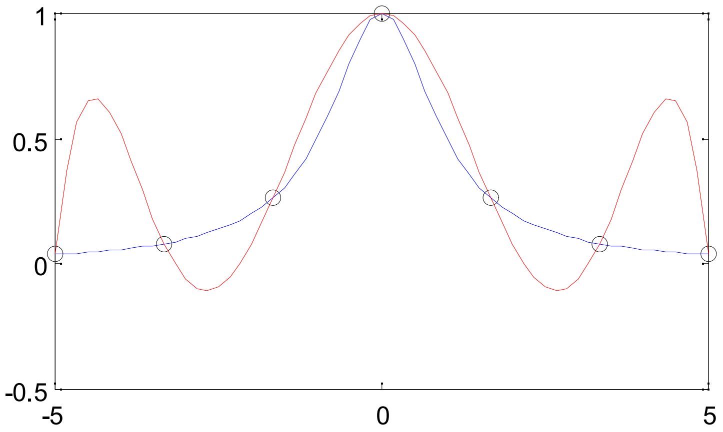
X	0	-0.4552	-0.6913	-0.8640	-0.9689	-0.9996
Y	0	0.3285	0.3467	0.2716	0.1408	-0.0160

2. 数据样条插值

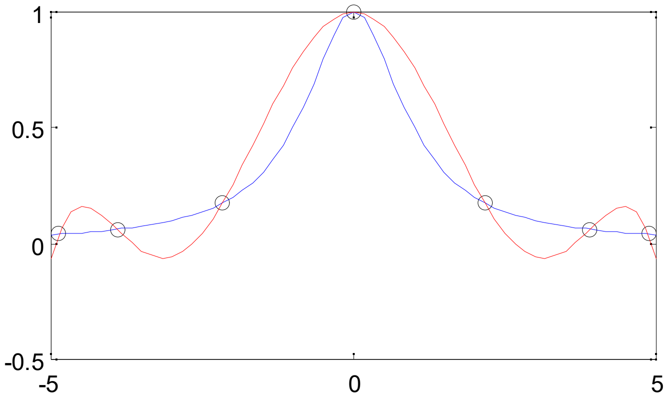
```
S=sqrt(diff(x).^2+diff(y).^2);
S=[0,S];Sk=cumsum(S);
sk=linspace(0,Sk(end),24);
xt=spline(Sk,x,sk);
yt=spline(Sk,y,sk);
```



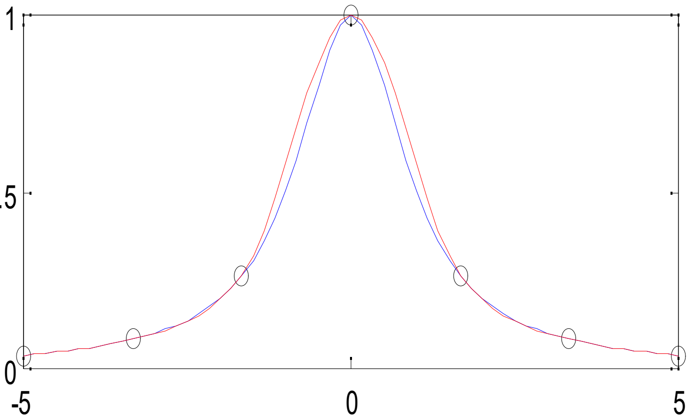
例2:龙格函数的插值逼近 $f(x)=\dfrac{1}{1+x^2}$



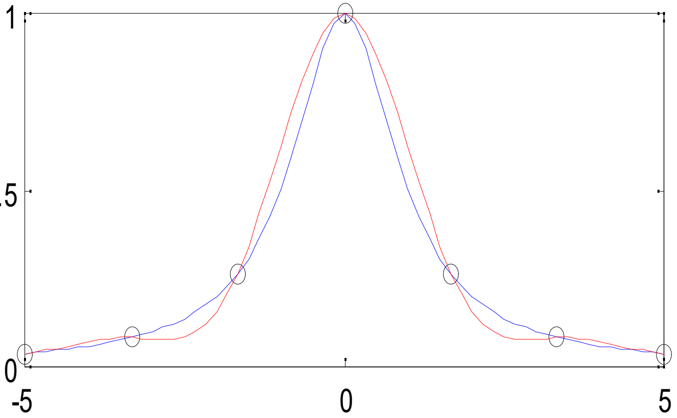
7结点等距插值



7结点切比雪夫插值



7结点埃尔米特插值



7结点样条插值

定义 5.4： 给定区间$[a, b]$上的一个分划：

$$a = x_0 < x_1 < \dots < x_n = b$$

已知 $f(x_j) = y_j \quad (j = 0,1,\cdots,n)$，如果

$$S(x) = \begin{cases} S_1(x), x \in [x_0, x_1] \\ S_2(x), x \in [x_1, x_2] \\ \cdots\cdots\cdots\cdots \\ S_n(x), x \in [x_{n-1}, x_n] \end{cases}$$

满足：(1) $S(x)$在 $[x_j, \ x_{j+1}]$上为三次多项式；

(2) $S''(x)$在区间$[a, b]$上连续；

(3) $S(x_j) = y_j \ (j = 0,1,\cdots,n)$.

则称 $S(x)$为三次样条插值函数.

用一阶导数表示的样条

n个三次多项式(每个三次多项式是4个待定系数), 待定系数共$4n$个！！

当$x\in[x_j, x_{j+1}]$ ($j=0,1,\ldots n-1$)时

$$S_j(x)=a_j+b_jx+c_jx^2+d_jx^3$$

由样条定义,可建立方程(4n-2)个！！ **Why?**

插值条件: $S(x_j)=y_j \quad (j=0,1,\cdots,n)$

连续性条件: $S(x_j+0)=S(x_j-0) \quad (j=1,\cdots,n-1)$

$$S'(x_j+0)=S'(x_j-0) \quad (j=1,\cdots,n-1)$$

$$S''(x_j+0)=S''(x_j-0) \quad (j=1,\cdots,n-1)$$

(1)自然边界条件: $S''(x_0)=0,\ S''(x_n)=0$

(2)周期边界条件: $S'(x_0)=S'(x_n),\ \ S''(x_0)=S''(x_n)$

(3)固定边界条件: $S'(x_0)=f'(x_0),\ \ S'(x_n)=f'(x_n)$

例 5.7: 已知$f(-1)=1, f(0)=0, f(1)=1$.验证下面分段三次多项式是自然样条函数.

$$S(x)=\begin{cases}\frac{1}{2}x^3+\frac{3}{2}x^2, & x\in[-1,0]\\ -\frac{1}{2}x^3+\frac{3}{2}x^2, & x\in[0,1]\end{cases}$$

证:显然 $S(-1)=1\quad S(1)=1\quad S(0-)=S(0+)=0$

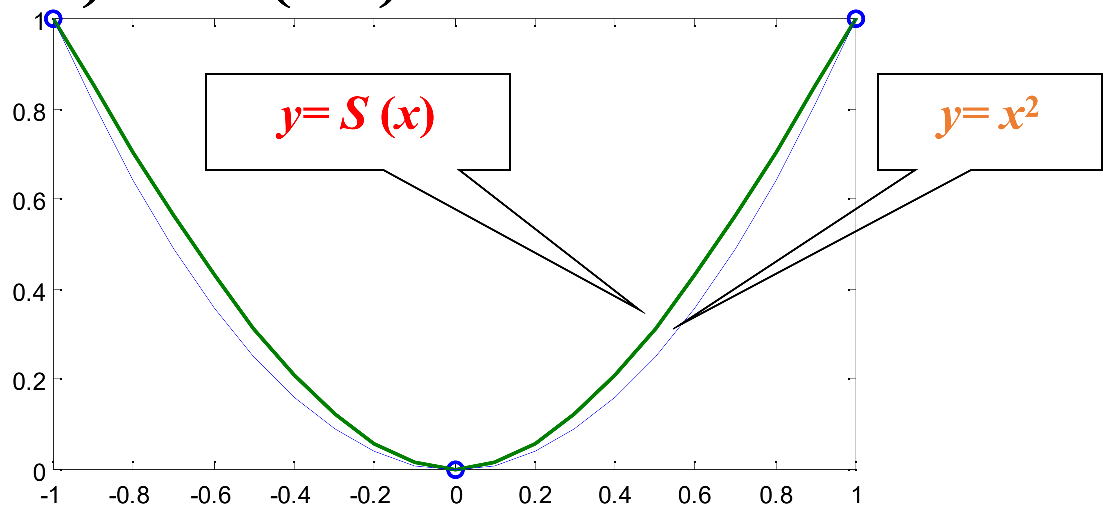
求导数得

$$S'(x)=\begin{cases}\frac{3}{2}x^2+3x,\ x\in[-1,0]\\ -\frac{3}{2}x^2+3x,\ x\in[0,1]\end{cases}\qquad S''(x)=\begin{cases}3x+3,\ x\in[-1,0]\\ -3x+3,\ x\in[0,1]\end{cases}$$

显然 $S'(-1)=-\frac{3}{2}$ $S'(1)=\frac{3}{2}$ $S'(0-)=S'(0+)=0$

$S''(-1)=0$ $S''(1)=0$ $S''(0-)=S''(0+)=3$

所以,$S(x)$是满足插值条件且**二阶导函数连续**的分段三次多项式



分段Hermite插值公式导出的三次样条方法

已知函数表

x	x_0	x_1	……	x_n
$f(x)$	y_0	y_1	……	y_n

设 $f(x)$ 在各插值节点 x_j 处的一阶导数为 m_j (未知)

取 $x_{j+1}-x_j=h$，$(j=0,1,2,\cdots,n)$.当 $x\in[x_j, x_{j+1}]$时，分段Hermite插值

$$S(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2 y_j+(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2 y_{j+1}$$

$$+(x-x_j)(\frac{x_{j+1}-x}{h})^2 m_j+(x-x_{j+1})(\frac{x-x_j}{h})^2 m_{j+1}$$

用一阶导数表示的样条

由$S''(x)$连续：有等式： $S''(x_j+0)=S''(x_j-0)$

考虑 $S''(x)$ 在区间$[x_j, x_{j+1}]$和$[x_{j-1}, x_j]$上表达式.

当 $x\in[x_j, x_{j+1}]$时, $S(x)$ 由基函数组合而成

$$\alpha_j(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2$$

$$\alpha_{j+1}(x)=(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2$$

$$\beta_j(x)=(x-x_j)(\frac{x_{j+1}-x}{h})^2$$

$$\beta_{j+1}(x)=(x-x_{j+1})(\frac{x-x_j}{h})^2$$

$\Rightarrow$

用一阶导数表示的样条

$$\begin{cases} \alpha_j''(x_j) = [\dfrac{-8}{h^3}(x_{j+1}-x) + (1+2\dfrac{x-x_j}{h})\dfrac{2}{h^2}]_{x=x_j} = -\dfrac{6}{h^2} \\ \alpha_{j+1}''(x_j) = [-\dfrac{8}{h^3}(x-x_j) + (1+2\dfrac{x_{j+1}-x}{h})\dfrac{2}{h^2}]_{x=x_j} = \dfrac{6}{h^2} \end{cases}$$

$$\begin{cases} \beta_j''(x_j) = [\dfrac{4}{h^2}(x-x_{j+1}) + (x-x_j)\dfrac{2}{h^2}]_{x=x_j} = -\dfrac{4}{h} \\ \beta_{j+1}''(x_j) = [\dfrac{4}{h^2}(x-x_j) + (x-x_{j+1})\dfrac{2}{h^2}]_{x=x_j} = -\dfrac{2}{h} \end{cases}$$

$$\Rightarrow \quad S''(x_j+0) = \alpha_j''(x_j)y_j + \alpha_{j+1}''(x_j)y_{j+1} + \beta_j''(x_j)m_j + \beta_{j+1}''(x_j)m_{j+1}$$

$$S''(x_j+0)=-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}$$

同理, 有

$$S''(x_j-0)=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

联立得:
$$-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

➔
$$m_{j-1}+4m_j+m_{j+1}=\frac{3}{h}(y_{j+1}-y_{j-1})$$
$(j=1, 2, \cdots\cdots, n\text{-}1)$

设自然边界条件成立, 即

$$S''(x_0+0)=-\frac{6}{h^2}y_0+\frac{6}{h^2}y_1-\frac{4}{h}m_0-\frac{2}{h}m_1=0$$

$$S''(x_n-0)=\frac{6}{h^2}y_{n-1}-\frac{6}{h^2}y_n+\frac{2}{h}m_{n-1}+\frac{4}{h}m_n=0$$

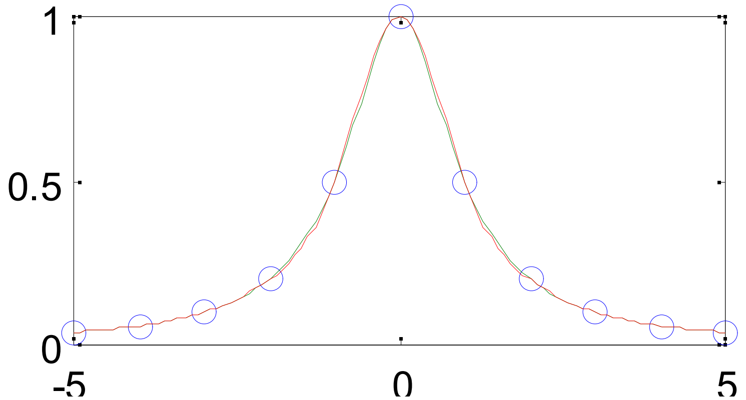
自然样条的导数值满足:

$$\boxed{2m_0+m_1=\frac{3}{h}[y_1-y_0]}\qquad \boxed{m_{n-1}+2m_n=\frac{3}{h}[y_n-y_{n-1}]}$$

$$\boxed{m_{j-1}+4m_j+m_{j+1}=\frac{3}{h}(y_{j+1}-y_{j-1})\qquad (\ j=1,2,\cdots\cdots,n\text{-}1\ )}$$

MATLAB样条命令：yi = spline(x, y, xi)

```
x=-5:5;y=1./(1+x.^2);
xi=-5:0.1:5;f=1./(1+xi.^2);
yi=spline(x,y,xi); error=max(abs(yi-f))
plot(x,y,'o',xi,f,xi,yi,'r')
```



error = 0.0220

曲率比较：计算公式 $K = \dfrac{|y''|}{(1+(y')^2)^{3/2}}$

样条插值函数的极性

设$f(x)\in C^2[a, b]$, 对于$a = x_0 < x_1 < \ldots < x_n = b$,有 $f(x_j)=y_j(j=0,1,\cdots,n)$.$S(x)$是满足$S(x_j)=y_j(j=0,1,\cdots,n)$ 的三次自然样条.则有

$$\|S''(x)\|\le\|f''(x)\|$$

证明: $\| f''(x) - S''(x) \|^2 = \int_a^b [f''(x) - S''(x)]^2 dx$

$$= \int_a^b [f''(x)]^2 dx - 2\int_a^b f''(x)S''(x)dx + \int_a^b [S''(x)]^2 dx$$

$$= \| f'' \|^2 \underline{-2\int_a^b [f''(x) - S''(x)]S''(x)dx} - \| S'' \|^2$$

$$\int_a^b [f''(x)-S''(x)]S''(x)dx = -\int_a^b [f'(x)-S'(x)]S'''(x)dx$$

$$= -\sum_{j=1}^{n} S_j'''[f(x)-S(x)]\Big|_{x_{j-1}}^{x_j} = 0$$

所以 $0 \le \| f''-S'' \|^2 = \| f'' \|^2 - \| S'' \|^2 \Rightarrow$

$$\| S'' \|^2 \le \| f'' \|^2$$

即 $$\int_a^b [S''(x)]^2 dx \le \int_a^b [f''(x)]^2 dx$$

样条函数S(x)在[a, b]上的总曲率最小.

学到了什么?

样条插值的算例

三次样条的概念

用一阶导数表示的样条

三次样条的极性