《数值分析》 13

主要内容：

- 切比雪夫插值结点
- 埃尔米特插值函数
- 分段插值函数

拉格朗日插值余项

$$R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi_n)}{(n+1)!}\omega_{n+1}(x)$$

其中， $\omega_{n+1}(x)=(x-x_0)(x-x_1)\cdots(x-x_n)$

目标/做什么：如何选取 $x_0,x_1,\cdots\cdots,x_n$ 更好？

（这里：切比雪夫多项式）

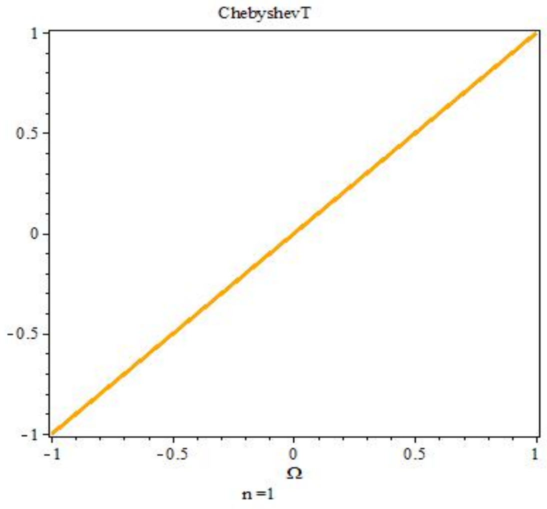
切比雪夫插值结点

$f(x) \in C[-1, 1]$, 令 $x = \cos\theta$, 则有 $[-1, 1] \leftarrow\rightarrow [0, \pi]$

将$g(\theta) = f(\cos\theta)$展开成余弦级数

$$g(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$$

百度百科中：正弦级数和余弦级数



n 阶切比雪夫多项式: $T_n = \cos(n\theta)$

$$\cos(n+1)\theta = 2\cos n\theta \cos\theta - \cos(n-1)\theta$$

$$T_{n+1} = 2xT_n - T_{n-1} \qquad (n = 1,2,\cdots)$$

$$T_0 = 1 \qquad T_1 = x$$

$$T_2 = 2x^2 - 1 \quad x_0^{(2)} = -\frac{1}{\sqrt{2}},\ x_1^{(2)} = \frac{1}{\sqrt{2}}$$

$$T_3 = 4x^3 - 3x \quad x_0^{(3)} = -\frac{\sqrt{3}}{2},\ x_1^{(3)} = 0,\ x_2^{(3)} = \frac{\sqrt{3}}{2}$$

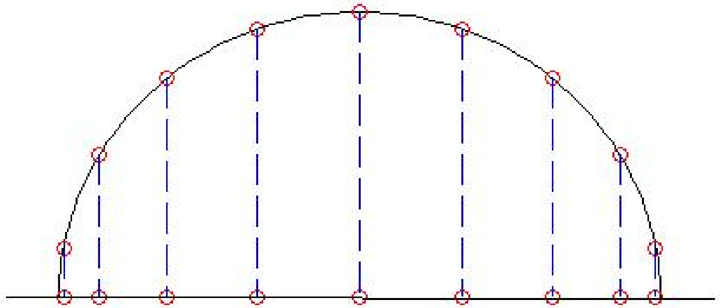
切比雪夫插值结点

$$\cos n\theta = 0 \rightarrow n\theta = \frac{(2k+1)\pi}{2} \rightarrow \theta = \frac{(2k+1)\pi}{2n}$$

$$\cos\theta = x \Leftrightarrow \theta = \arccos x$$

$$\rightarrow \arccos x = \frac{(2k+1)\pi}{2n} \rightarrow x_k = \cos(\frac{(2k+1)\pi}{2n})$$

$(k=0,1,\cdots,n-1)$



n次多项插值的切比雪夫结点

$$\rightarrow x_k = \cos(\frac{(2k+1)\pi}{2(n+1)})$$

$(k=0,1,\cdots,n)$

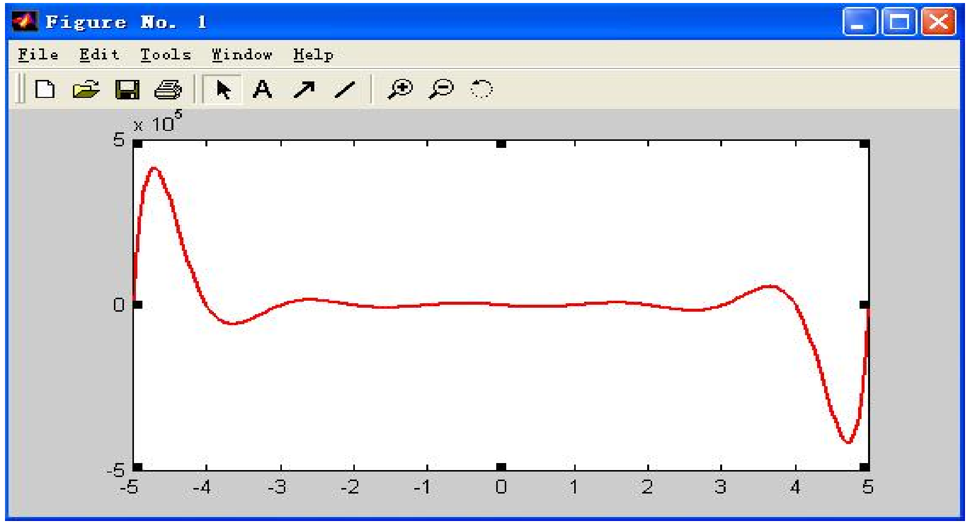
例1. 函数 $f(x)=\dfrac{1}{1+x^2}$ $x\in[-5,5]$

取等距插值结点： -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5

$$\boxed{f(x)=L_{10}(x)+\frac{f^{(11)}(\xi_n)}{11\,!}\omega_{11}(x)}$$

$\omega_{11}(x)=(x+5)(x+4)(x+3)(x+2)(x+1)x(x-1)(x-2)(x-3)(x-4)(x-5)$

$\omega_{11}(x)$ →



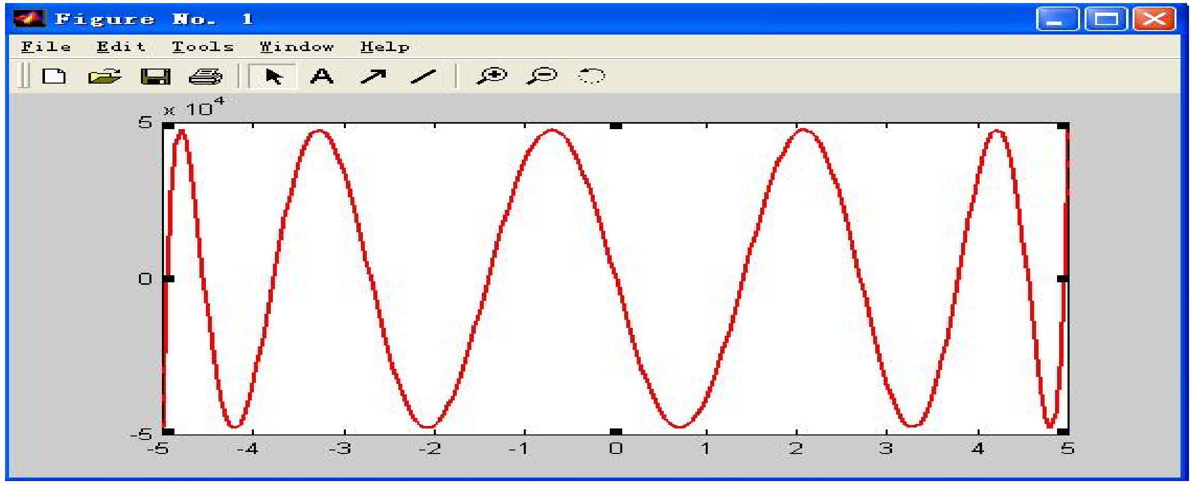
在[-5, 5]区间上,取11个切比雪夫结点

$$x_k = 5\cos\left(\frac{(2k+1)\pi}{22}\right) \qquad (k=10, 9, 8, \cdots, 1, 0)$$

-4.9491	-4.5482	-3.7787	-2.7032	-1.4087	0.0000	1.4087
2.7032	3.7787	4.5482	4.9491			

$$\omega_{11}(x)=(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{10})$$

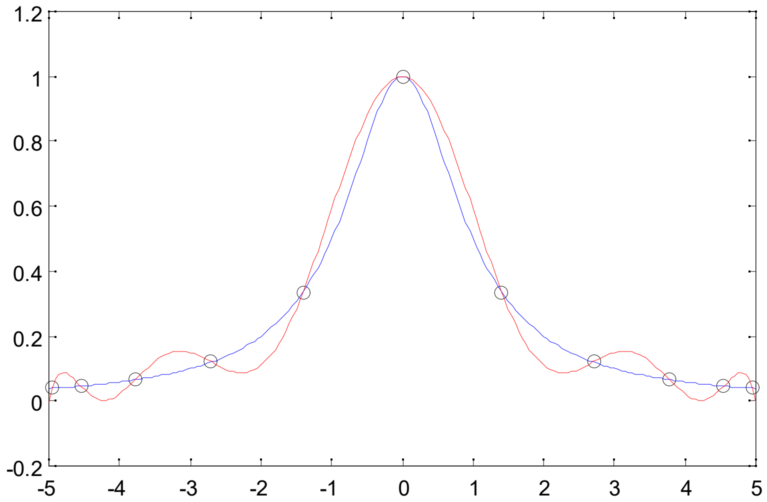
$\omega_{11}(x) \rightarrow$



切比雪夫插值结点

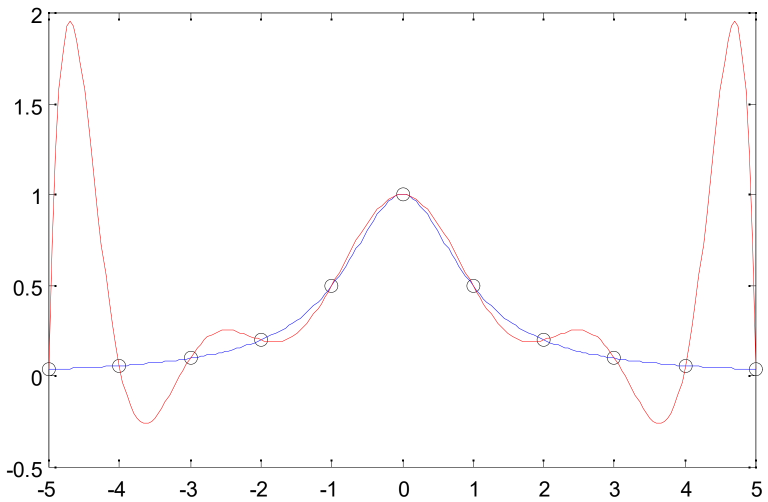
插值函数$L_{10}(x)$取

切比雪夫结点插值



插值函数$L_{10}(x)$取

等距结点插值



插值条件中除函数值插值条件外，还有导数值插值条件，即:

已知2n+2个条件

x_i	x_0	x_1	$\cdots$	x_n
$y_i=f(x_i)$	y_0	y_1	$\cdots$	y_n
$y_i'=f'(x_i)$	y_0'	y_1'	$\cdots$	y_n'

求:一个次数不超过2n+1的多项式$H_{2n+1}(x)$

三次Hermite插值问题　设 $f(x) \in C^4[x_0, x_1]$

已知在插值节点x_0和x_1的函数值和导数值为：

$$f(x_0) = y_0 \qquad f(x_1) = y_1$$
$$f'(x_0) = m_0 \qquad f'(x_1) = m_1$$

可以求到次数为3次的多项式$H_3(x)$, 称为三次Hermite 插值多项式

插值条件：

$$H(x_0) = y_0 \quad H(x_1) = y_1$$
$$H'(x_0) = m_0 \quad H'(x_1) = m_1$$

采用基函数方式构造*H*(*x*)：

$$\boxed{H(x)=\alpha_0(x)y_0+\alpha_1(x)y_1+\beta_0(x)m_0+\beta_1(x)m_1}$$

插值条件：

$$\boxed{\begin{array}{ll} H(x_0)=y_0 & H(x_1)=y_1 \\ H'(x_0)=m_0 & H'(x_1)=m_1 \end{array}}$$

插值条件表

	函数值		导数值	
	x_0	x_1	x_0	x_1
$\alpha_0(x)$	1	0	0	0
$\alpha_1(x)$	0	1	0	0
$\beta_0(x)$	0	0	1	0
$\beta_1(x)$	0	0	0	1

如何求 $\alpha_0(x)$ $\alpha_1(x)$ $\beta_0(x)$ $\beta_1(x)$

$\boldsymbol{\alpha_0(x)}$

由于 $\alpha_0(x_1)=\alpha_0'(x_1)=0$, 故 $\alpha_0(x)$含有 $(x-x_1)^2$ 因子。可设

$$\alpha_0(x)=(a+b(x-x_0))(x-x_1)^2$$

其中a，b为待定系数。

由 $\alpha_0(x_0)=1$, 可得 $a=\dfrac{1}{(x_0-x_1)^2}$.

由 $\alpha_0'(x_0)=0$, 可得 $b=\dfrac{-2}{(x_0-x_1)^3}$.

将a，b代入得

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2$$

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2$$

类似地，将 x_0, x_1 互换，可得

$$\alpha_1(x)=\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

$\boxed{\beta_0(x)}$

由于$\beta_0(x_0)=\beta_0(x_1)=\beta_0'(x_1)=0$,故$\beta_0(x)$含有$(x-x_0)(x-x_1)^2$因子。可设

$$\beta_0(x)=c(x-x_0)(x-x_1)^2$$

其中c为待定系数。

由 $\beta_0'(x_0)=1$,可得 $c=\dfrac{1}{(x_0-x_1)^2}$.

$$\boxed{\beta_0(x)=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2}$$

类似地，将 x_0,x_1 互换，可得

$$\boxed{\beta_1(x)=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2}$$

埃尔米特插值

最终求得所有4个基函数（针对三次Hermite插值）

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right) \qquad 2\,\beta_0(x)=(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2$$

$$\alpha_1(x)=\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2 \qquad \beta_1(x)=(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

代入4个基函数即可得：三次Hermite插值多项式

$$\boxed{H(x)=\alpha_0(x)y_0+\alpha_1(x)y_1+\beta_0(x)m_0+\beta_1(x)m_1}$$

定理：两点三次Hermite插值的误差估计式

$$R(x)=f(x)-H_3(x)=\frac{f^{(4)}(\xi)}{4!}[(x-x_0)(x-x_1)]^2$$

证明: 由插值条件知

$$R(x_0)=R'(x_0)=0 \qquad R(x_1)=R'(x_1)=0$$

取 x 异于 x_0 和 x_1, 有

$$R(x)=C(x)(x-x_0)^2(x-x_1)^2$$

利用 $f(x)-H(x)=C(x)(x-x_0)^2(x-x_1)^2$

构造辅助函数

$$F(t)=f(t)-H(t)-C(x)(t-x_0)^2(t-x_1)^2$$

显然,$F(t)$ 有三个零点 x_0, x, x_1, 由Roll定理知,存在 $F'(t)$ 两个零点 t_0, t_1. 故 $F'(t)$ 有四个相异零点

$$x_0 < t_0 < x < t_1 < x_1$$

反复应用 Roll 定理, 得 $F^{(4)}(t)$ 知一个零点设为ξ

$$F(t) = f(t) - H(t) - C(x)(t-x_0)^2(t-x_1)^2$$

$$\Rightarrow \quad F^{(4)}(\xi) = f^{(4)}(\xi) - C(x)(4!) = 0$$

$$\Rightarrow \quad C(x) = \frac{f^{(4)}(\xi)}{4!}$$

$$\boxed{R(x) = \frac{f^{(4)}(\xi)}{4!}[(x-x_0)(x-x_1)]^2}$$

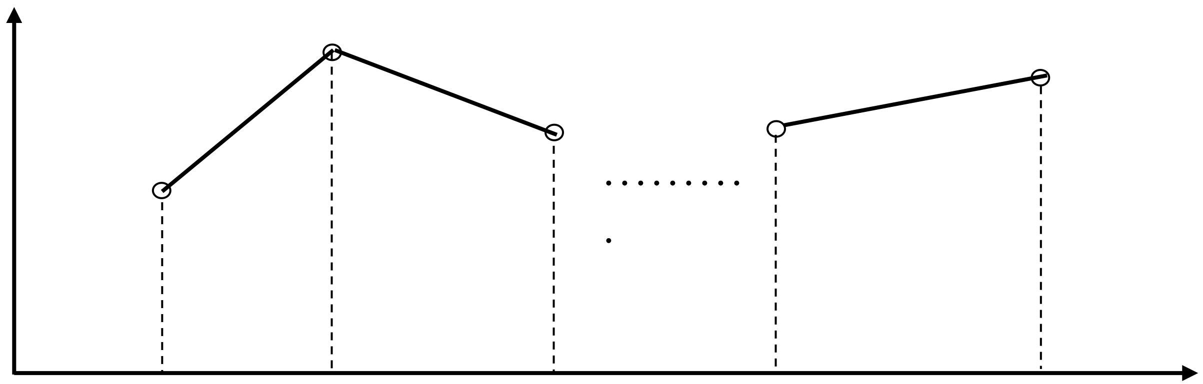
分段线性插值

插值节点满足：$x_0<x_1<\cdots\cdots<x_n$ 已知

$$y_j=f(x_j) \qquad (j=0,1,2,\cdots,n)$$

$x\in[x_j, \ x_{j+1}]$时, 线性插值函数

$$L_h(x)=\frac{x_{j+1}-x}{x_{j+1}-x_j}y_j+\frac{x-x_j}{x_{j+1}-x_j}y_{j+1} \quad (j=0,1,\cdots,n\text{-}1)$$



分段三次Hermite插值

已知函数值和导数值 $y_j = f(x_j),\ m_j = f'(x_j)$

$$H_h(x) = (1+2\frac{x-x_j}{x_{j+1}-x_j})(\frac{x_{j+1}-x}{x_{j+1}-x_j})^2 y_j \quad (j=0,1,2,\cdots,n)$$

$$+(1+2\frac{x_{j+1}-x}{x_{j+1}-x_j})(\frac{x-x_j}{x_{j+1}-x_j})^2 y_{j+1}$$

$$+(x-x_j)(\frac{x_{j+1}-x}{x_{j+1}-x_j})^2 m_j + (x-x_{j+1})(\frac{x-x_j}{x_{j+1}-x_j})^2 m_{j+1}$$

$x \in [x_j,\ x_{j+1}]$ $(j=0,1,2,\cdots,n\text{-}1)$

学到了什么？

切比雪夫插值结点

埃尔米特插值

分段插值函数