



## 5.2 矩阵的相似对角化

主要内容：相似矩阵的基本概念

矩阵的相似对角化

## 一. 相似矩阵的基本概念

例 设矩阵

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -5 & 1 \end{pmatrix}$$

求  $A^{10}$ .

解  $A = P\Lambda P^{-1}$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

$$A^{10} = P\Lambda P^{-1}P\Lambda P^{-1} \dots P\Lambda P^{-1}P\Lambda P^{-1} = P\Lambda^{10}P^{-1}$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} (-2)^{10} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

问题: 1. 什么样的矩阵有这样的  $P$  与  $\Lambda$ ?

2.  $P = ?$   $\Lambda = ?$

## 一. 矩阵相似的定义与性质

定义 设  $A$  与  $B$  都是  $n$  阶矩阵, 如果存在可逆矩阵  $P$ , 使

$$P^{-1}AP = B$$

则称  $A$  与  $B$  相似, 记为  $A \sim B$ .

简单性质: (1) 反身性  $A \sim A$ ;

(2) 对称性  $A \sim B \Rightarrow B \sim A$ ;

(3) 传递性  $A \sim B$  且  $B \sim C \Rightarrow A \sim C$ .

证(3):  $A = PBP^{-1}$   $B = QCQ^{-1}$

$$A = PQCQ^{-1}P^{-1} = DCD^{-1} \quad (D = PQ).$$

## 定理1 相似矩阵有相同的特征值 .

证 设  $A \sim B$ , 则  $B = P^{-1}AP$  .

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}| |\lambda I - A| |P| \\ &= |\lambda I - A| \end{aligned}$$

思考：相似矩阵有相同的行列式？

## 二. 矩阵的相似对角化

定理2 设矩阵

$$A \sim \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

则  $\lambda_1, \lambda_2, \dots, \lambda_n$  是  $A$  的全部特征值.

$$\begin{aligned} \text{证: } |\lambda I - A| &= \begin{vmatrix} \lambda - \lambda_1 & & & \\ & \lambda - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda - \lambda_n \end{vmatrix} \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \end{aligned}$$

- $\therefore \Lambda$  的全部特征值是:  $\lambda_1, \lambda_2, \dots, \lambda_n$ .
- $\therefore A$  与  $\Lambda$  的特征值相同,
- $\therefore A$  的全部特征值是:  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

**定理3**  $n$ 阶矩阵 $A$ 与对角矩阵相似的充分必要条件是 $A$ 有 $n$ 个线性无关的特征向量.

证:充分性 设 $A$ 有 $n$ 个线性无关的特征向量:

$$P_1, P_2, \dots, P_n \quad AP_i = \lambda_i P_i \quad (i = 1, 2, \dots, n)$$

$$\text{令 } P = (P_1 \ P_2 \ \dots \ P_n) \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$\text{则 } AP = P\Lambda \quad P^{-1}AP = \Lambda$$

$$\therefore A \sim \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$



必要性 设  $P^{-1}AP = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

则  $AP = P\Lambda$

设  $P = (P_1 \ P_2 \ \cdots \ P_n)$

则  $(AP_1 \ AP_2 \ \cdots \ AP_n) = (\lambda_1 P_1 \ \lambda_2 P_2 \ \cdots \ \lambda_n P_n)$

$AP_i = \lambda_i P_i \quad (i = 1, 2, \dots, n)$

$\therefore P_1, P_2, \dots, P_n$  是  $A$  的  $n$  个线性无关的特征向量.

**定理4** 矩阵  $A$  不同特征值的特征向量线性无关 .

证: 设  $A\alpha_1 = \lambda_1\alpha_1, A\alpha_2 = \lambda_2\alpha_2, \dots, A\alpha_m = \lambda_m\alpha_m$ , 且  $\lambda_1, \lambda_2, \dots, \lambda_m$  互不相同.

当  $m = 2$  时, 设  $k_1\alpha_1 + k_2\alpha_2 = \mathbf{0}$ . (1)

则  $A(k_1\alpha_1 + k_2\alpha_2) = k_1A\alpha_1 + k_2A\alpha_2 = k_1\lambda_1\alpha_1 + k_2\lambda_2\alpha_2 = \mathbf{0}$  (2)

又由式(1):  $k_1\lambda_1\alpha_1 + k_2\lambda_1\alpha_2 = \mathbf{0}$  (3)

(2)-(3):  $k_2(\lambda_2 - \lambda_1)\alpha_2 = \mathbf{0}$

$\because \lambda_1 \neq \lambda_2$  且  $\alpha_2 \neq \mathbf{0}$ ,  $\therefore k_2 = \mathbf{0}$ , 同理,  $k_1 = \mathbf{0}$ ,

$\therefore \alpha_1, \alpha_2$  线性无关.

**推论1** 如果矩阵  $A$  的特征值都是单特征根, 则  $A$  与对角矩阵相似 .

证: 设  $\lambda_1, \lambda_2, \dots, \lambda_n$  是  $A$  的互异特征值:

$\alpha_1, \alpha_2, \dots, \alpha_n$  是它们对应的特征向量

则  $\alpha_1, \alpha_2, \dots, \alpha_n$  线性无关,

$\therefore A$  与对角矩阵相似.

**推论2** 设  $\lambda_1, \lambda_2, \dots, \lambda_k$  是矩阵  $A$  的不同特征值,  
 $\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{ir_i}$  是  $\lambda_i$  的线性无关特征向量,  
则  $\alpha_{11}, \dots, \alpha_{1r_1}, \dots, \alpha_{k1}, \dots, \alpha_{kr_k}$  线性无关 .

**推论3**  $n$ 阶矩阵 $A$ 与对角矩阵相似

$\Leftrightarrow$  若 $\lambda_i$ 是 $A$ 的 $k_i$ 重特征值, 则 $(\lambda_i I - A)X = \mathbf{0}$ 的基础解系由 $k_i$ 个解向量组成.

$\Leftrightarrow R(\lambda_i I - A) = n - k_i$ .

分析: 设 $\lambda_1, \lambda_2, \dots, \lambda_r$ 是 $A$ 的全部互异特征值,

则  $k_1 + k_2 + \dots + k_r = n$ .

例1 设矩阵  $A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$ , 求  $A^{10}$ .

$$\text{解 } |\lambda I - A| = \begin{vmatrix} \lambda - 4 & -6 & 0 \\ 3 & \lambda + 5 & 0 \\ 3 & 6 & \lambda - 1 \end{vmatrix} = (\lambda + 2)(\lambda - 1)^2$$

$$\lambda_1 = -2, \lambda_2 = 1 \text{ (二重)}.$$

$$(\lambda_1 I - A) = \begin{pmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \\ 3 & 6 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x}_1 = -\mathbf{x}_3, \mathbf{x}_2 = \mathbf{x}_3, \boldsymbol{\alpha}_1 = (-1, 1, 1)^T$$

$$\lambda_2 I - A = \begin{pmatrix} -3 & -6 & 0 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = -2x_2 + 0x_3$$

$$\alpha_2 = (-2, 1, 0)^T, \quad \alpha_3 = (0, 0, 1)^T.$$

$$\text{令 } P = (\alpha_1 \quad \alpha_2 \quad \alpha_3) = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\text{则 } P^{-1}AP = \begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \Lambda$$

$$A = PAP^{-1}$$

$$A^{10} = PA^{10}P^{-1}$$

$$= \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1024 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1022 & -2046 & 0 \\ 1023 & 2047 & 0 \\ 1023 & 2046 & 1 \end{pmatrix}$$

例2 设矩阵

$$A = \begin{pmatrix} a & a & \cdots & a \\ a & a & \cdots & a \\ \cdots & \cdots & \cdots & \cdots \\ a & a & \cdots & a \end{pmatrix} \quad (a \neq 0)$$

求  $A$  的特征值与特征向量，  
并判断  $A$  能否与对角矩阵相似。

$$\text{解 } |\lambda I - A| = \begin{vmatrix} \lambda - a & -a & \cdots & -a \\ -a & \lambda - a & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & \lambda - a \end{vmatrix}$$



# 矩阵的相似

$$= \begin{vmatrix} \lambda - na & \lambda - na & \cdots & \lambda - na \\ -a & \lambda - a & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & \lambda - a \end{vmatrix} = (\lambda - na) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -a & \lambda - a & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & \lambda - a \end{vmatrix}$$

$$= (\lambda - na) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= (\lambda - na)\lambda^{n-1},$$

$$\lambda_1 = na, \quad \lambda_2 = 0 \quad (n-1 \text{重}).$$

$$(\lambda_1 I - A)X = 0 \quad \text{即}$$

$$\begin{pmatrix} (n-1)a & -a & \cdots & -a \\ -a & (n-1)a & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & (n-1)a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\alpha_1 = (1, 1, \cdots, 1)^T$$

$\lambda_1$  对应的特征向量为 :  $k_1 \alpha_1$  ( $k_1 \neq 0$ ).

$$\lambda_2 I - A = \begin{pmatrix} -a & -a & \cdots & -a \\ -a & -a & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & -a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$x_1 + x_2 + \cdots + x_n = 0$$

$$\alpha_2 = (1, -1, 0, \cdots, 0, 0)^T,$$

$$\alpha_3 = (0, 1, -1, \cdots, 0, 0)^T,$$

... ..

$$\alpha_n = (0, 0, 0, \cdots, 1, -1)^T.$$

**$A$  有  $n$  个线性无关的特征向量, 能与对角矩阵相似.**

例3 下列矩阵能否与对角矩阵相似 .

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$$

$$\text{解 } |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & 2 \\ 2 & 2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda + 1)(\lambda - 3)$$

$$A \sim \text{diag}(1, -1, 3)$$

$$|\lambda I - B| = \begin{vmatrix} \lambda - 3 & 1 & 2 \\ -2 & \lambda & 2 \\ -2 & 1 & \lambda + 1 \end{vmatrix} = \lambda(\lambda - 1)^2$$

$$\lambda_1 = 0, \quad \lambda_2 = 1 \text{ (二重)}.$$

$$\lambda_2 I - B = \begin{pmatrix} -2 & 1 & 2 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = \frac{1}{2}x_2 + x_3, \quad \alpha_2 = (1, 2, 0)^T, \quad \alpha_3 = (1, 0, 1)^T.$$

$$B \sim \text{diag}(0, 1, 1)$$

$$\text{又 } R(\lambda_2 I - B) = 1,$$

$\therefore B$ 与对角矩阵相似.

$$|\lambda I - C| = \begin{vmatrix} \lambda - 3 & -1 & 0 \\ 4 & \lambda + 1 & 0 \\ -4 & 8 & \lambda + 2 \end{vmatrix} = (\lambda - 1)^2 (\lambda + 2)$$

$$\lambda_1 = 1, (\text{二重}) \quad \lambda_2 = -2,$$

$$R(\lambda_1 - C) = 2,$$

$\therefore C$ 不能与对角矩阵相似.

例4 设  $A = \begin{pmatrix} 0 & 0 & 1 \\ x & 1 & y \\ 1 & 0 & 0 \end{pmatrix} \sim \Lambda$  为对角阵 .

求  $x$  与  $y$  应满足的条件 .

$$\text{解: } |\lambda I - A| = \begin{vmatrix} \lambda & 0 & -1 \\ -x & \lambda - 1 & -y \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda - 1)^2 (\lambda + 1)$$

$$\lambda_1 = 1 \text{ (二重)}, \lambda_2 = -1.$$

$A \sim \text{对角阵} \Leftrightarrow \lambda_1$  有两个线性无关的特征向量

$$\Leftrightarrow R(\lambda_1 E - A) = 1$$

$$\lambda_1 E - A = \begin{pmatrix} 1 & 0 & -1 \\ -x & 0 & -y \\ -1 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -x-y \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(\lambda_1 E - A) = 1 \Leftrightarrow -x - y = 0$$

$$\text{即 } x + y = 0.$$



例5 设  $A \sim B$ ,  $C \sim D$ , 证明:

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \sim \begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix} .$$

证:  $\because A \sim B, C \sim D$

$\therefore \exists$  可逆矩阵  $P, Q$ , 使

$$P^{-1}AP = B, Q^{-1}CQ = D$$

$$\begin{pmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & Q^{-1} \end{pmatrix} \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & Q \end{pmatrix} = \begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}$$

$$\therefore \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \sim \begin{pmatrix} C & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix} .$$

**例6** 设  $A$  是 3 阶矩阵且  $I+A, 3I-A, I-3A$  均不可逆. 证明:

(1)  $A$  可逆, (2)  $A$  与对角矩阵相似.

证 (1)  $\because I+A$  不可逆,  $\therefore |I+A|=0$ ,

$$\therefore (-1)^3 |-I-A|=0 \Rightarrow |-I-A|=0,$$

$\therefore \lambda_1 = -1$  是  $A$  特征值.

$\lambda_2 = 3$  是  $A$  的特征值.

$$|I-3A|=3^3 \left| \frac{1}{3}I-A \right|=0 \Rightarrow \left| \frac{1}{3}I-A \right|=0,$$

$\therefore \lambda_3 = \frac{1}{3}$  是  $A$  的特征值.  $A$  的特征值均不为零, 故  $A$  可逆.

(2)  $\because$   $A$ 的特征值都是单特征值，

$$\therefore A \sim \Lambda = \begin{pmatrix} -1 & & \\ & 3 & \\ & & \frac{1}{3} \end{pmatrix} .$$

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矩阵的相似对角化